Why do I keep arguing with you?

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Abstract

Dear Aric, I have discovered that there have been some inconveniences in our latest discussions, thus, I want to verify them. Here I would like to clarify my point in the most convenient way. As we are both scientists, I thought that our argument would only benefit from the clarification of conventions we both use. In particular, it seems that we understand limits differently. I want to show you, why am I providing you with those counter-examples with those tricky sequences and why they still should generate the same function, as you desire. Moreover, I shall make a remark that your difference of series does not exist everywhere else, except the critical line with your sequence, which makes your SSE with the restriction to certain summation method to be just a reformulation of the Riemann hypothesis, which cannot be considered as a proof. Later I shall show how this difference of series changes with different summation methods (which was exactly the point of the commenter under your video about independent limits and that is the case, which should have been proven impossible to make your argument valid). Moreover, while surfing some old books on the topic I met some formulas, which are similar to yours, but still haven't been used in the way you suggest. Why? I want to explain by the current paper.

1 Introduction

1.1 The definition of limit

First of all, I would like to remind the definition of limit. We say that for some sequence $\{a_k | k \geq 1\}$ a is a limit of this sequence (i.e., $\lim_{k\to\infty} a_k = a$), if for any given number $\epsilon > 0$ there exists an integer N such that for any $k > N$ it holds that

$$|a_k - a| < \epsilon. \tag{1}$$

Here by $|.|$ we denote the metric, which we equip our sequence with. Indeed, we can choose different metric properties for different objects, but we shall mostly work with the case, when our metric is just an absolute value. I shall keep the numeration even for this definition to make a reference in further discussion. As you can see, the speed of convergence is not relevant, if the limit exists. In other words, the notion of limit should not be related to the sequence as a function of natural numbers, but it is a property of the sequence as a set. We shall return to this discussion in the next section, but for now I would like to reformulate the limit definition to make it more convenient.

Lemma 1. *Let $A = \{a_k | k \geq 1\}$ be a sequence. Then the statement $\lim_{k\to\infty} a_k = a$ is equivalent to the statement that a is the only accumulation point for any infinite subset of A.*

Proof. Suppose that $\lim_{k\to\infty} a_k = a$. Then for choosing an infinite subset of A it is enough to choose a sequence of indexes of elements of A, since each index corresponds to a certain element of the sequence. Denote this choice of indexes as $\{m(k)|k \geq 1\}$, which is infinite as we discuss the infinite subset. Without the loss of generality assume that this set of indexes is numerated in a strictly ascending order since reindexing does not change a subset. As $\{m(k)|k \geq 1\}$ is infinite, it grows infinitely. Then by choosing an ascending order for any $\epsilon > 0$ we are able to choose a natural number k_0 such that $m(k_0) > N$ in terms of definition (1). By the choice of our sequence of indexes we obtain that $\forall k > k_0$ it holds that $m(k) > m(k_0)$ and by the definition (1) we obtain the desired implication by the definition of accumulation point, i.e., for any infinite subset of A a is the only accumulation point. The uniqueness is obtained from the uniqueness of the limit.

The required calculation is provided bellow. If we assume that there exists another limit $b \neq a$, we obtain that for arbitrary $\epsilon > 0$ we obtain by using the triangular inequality that for $k > N$

$$2\epsilon > |a_k - a| + |a_k - b| > |a - b|. \tag{2}$$

Here we have the contradiction as $\epsilon > 0$ is chosen arbitrary and $a \neq b$ implies that there exists a positive distance between them.

Conversely, if we suppose that a is the unique accumulation point for any infinite subset, we obtain the definition (1) by indexing the elements of chosen subset. □

1.2 The possible extensions of limit definition

As one can expect, the extension of some definition allows us to apply the certain concept, where it is traditionally not applicable. But one condition should be respected: the extended definition MUST work in the same way with objects, for which it is applicable. Let us look at the most popular limit extensions. Of course, they are summation methods, but the limit of any sequence, if it exists, $\{a_k|k \geq 1\}$ we can understand as the sum of the series $a_1 + \sum_{k=1}^{\infty}(a_{k+1} - a_k)$. We prove it as the next lemma to refer to it later.

Lemma 2. *Suppose that $\{a_k|k \geq 1\}$ is convergent. Then the following equality holds.*

$$\lim_{k\to\infty} a_k = a_1 + \sum_{n=1}^{\infty}(a_{n+1} - a_n).$$

Proof. As the sum of the series is a limit of its partial sums by definition, let us use this definition to obtain the lemma statement.

$$a_1 + \sum_{n=1}^{\infty}(a_{n+1} - a_n) = \lim_{k\to\infty}(a_1 + \sum_{n=1}^{k}(a_{n+1} - a_n))$$

$$\lim_{k\to\infty}(a_1 + (a_2 - a_1) + (a_3 - a_2) + \cdots + (a_{k+1} - a_k) = \lim_{k\to\infty} a_{k+1}.$$

By (1) $\lim_{k\to\infty} a_{k+1} = \lim_{k\to\infty} a_k$, which completes the proof. □

As we have proven that any extension of summation generates the limit extension, you can see that it works for Cezaro summation (`https://en.wikipedia.org/wiki/Ces%C3%A0ro_summation`) and for Abel summation (`https://encyclopediaofmath.org/wiki/Abel_summation_method`). This is the difference between extension and substitution.

1.3 ABC zeta-function and the Identity theorem

You know how fascinated with ABC zeta-function I am, so here I present a proof of mine for ABC zeta-function representing the Riemann zeta-function with the usage of Identity theorem. One of the purposes of mine is to show how Identity theorem actually works and to remove one inconvenience with its understanding.

Theorem 1. *(Identity theorem, `https://en.wikipedia.org/wiki/Identity_theorem`) Let f and g be two holomorphic functions defined on the domain D (the domain is open and connected set). Suppose that $f = g$ on some subset $S \subset D$ and S has got an accumulation point. Then $f = g$ identically on D.*

Remark 1. *Connectivity is important. Indeed, by the way of contradiction let us assume that it is not. Then let us define the domain D as a union of two disjoint domains: $D = B_1(0) \cup B_1(4) =: D_1 \cup D_2$. Then let us define two functions: $f = 1$ on D and $g = 1$ on D_1, but $g = 2$ on D_2. We note that f and g are both holomorphic as they both satisfy the Cauchy-Riemann equations at each point of D. If we drop the condition of connectivity, we can see that $f = g$ on D_1, which consists of accumulation points. Thus, we could deduce that $f = g$ on D and therefore, $f = g$ on D_2 as well. But then we obtain that $1 = 2$, which is a contradiction. So we MUST have the connectivity of the set we discuss.*

And when we have the Identity theorem clarified, we are ready to show that ABC zeta-function indeed generates the analytic continuation of Riemann zeta-function.

Theorem 2. *Let $s \in \{z | \Re(s) > 0, s \neq 1\}$. Then for any $b \in \mathbb{N}$ the following identity holds.*

$$\zeta(s) = \sum_{n=1}^{b} \frac{1}{n^s} - \frac{b^{1-s}}{1-s} - s\int_b^{\infty} \frac{x - \lfloor x \rfloor}{x^{s+1}} dx.$$

Proof. The domain $\{z|\Re(s) > 0, s \neq 1\} =: D$ is obviously connected. Let us check that $\sum_{n=1}^{b} \frac{1}{n^s} - \frac{b^{1-s}}{1-s} - s\int_b^{\infty} \frac{x-\lfloor x \rfloor}{x^{s+1}} dx$ is holomorphic on $s \in \{z|\Re(s) > 0, s \neq 1\}$. $\sum_{n=1}^{b} \frac{1}{n^s}$ is holomorphic on D as a finite sum of exponents, $\frac{b^{1-s}}{1-s}$ is holomorphic on D as the product of exponent and linear-fractional function with the pole outside of D and $s\int_b^{\infty} \frac{x-\lfloor x \rfloor}{x^{s+1}} dx$ is holomorphic as the product of linear function and the absolutely convergent in the neighborhood of any s integral of holomorphic with respect to s function. Now it remains to show that $\sum_{n=1}^{b} \frac{1}{n^s} - \frac{b^{1-s}}{1-s} - s\int_b^{\infty} \frac{x-\lfloor x \rfloor}{x^{s+1}} dx$ coincides with the Riemann zeta-function for $\Re(s) > 1$ and the claim is proven by Identity theorem. Set

$$\sum_{n=1}^{b} \frac{1}{n^s} - \frac{b^{1-s}}{1-s} - s\int_b^{\infty} \frac{x-\lfloor x \rfloor}{x^{s+1}} dx =: A + B + C. \tag{3}$$

Consider C on $\Re(s) > 1$. We obtain the following.

$$C = -s\int_b^{\infty} \frac{x-\lfloor x \rfloor}{x^{s+1}} dx = -s\int_b^{\infty} \frac{dx}{x^s} + \sum_{n=b}^{\infty} \int_n^{n+1} \frac{n}{x^{s+1}} dx$$

$$= s\frac{b^{1-s}}{1-s} + s\frac{1}{(-s)} \sum_{n=1}^{\infty} [\frac{n}{(n+1)^s} - \frac{n}{n^s}] = s\frac{b^{1-s}}{1-s} + b^{1-s} + \sum_{n=b+1} \frac{1}{n^s}.$$

Indeed, here we noticed that the series is "telescoping": in each next term we add $n+1$ terms of the kind $\frac{1}{(n+1)^s}$ while n of those terms were subtracted by the previous term. Now it remains to substitute this into (3) and complete the proof.

$$\sum_{n=1}^{b} \frac{1}{n^s} - \frac{b^{1-s}}{1-s} - s\int_b^{\infty} \frac{x-\lfloor x \rfloor}{x^{s+1}} dx = \sum_{n=1}^{b} \frac{1}{n^s} - \frac{b^{1-s}}{1-s} + s\frac{b^{1-s}}{1-s} + b^{1-s} + \sum_{n=b+1} \frac{1}{n^s}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^s} - \frac{b^{1-s}}{1-s} + \frac{b^{1-s}}{1-s} = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

which is precisely the definition of Riemann zeta-function for $\Re(s) > 1$. Thus, the proof is complete.

□

Remark 2. *Here we show, how the analytic continuation should be proven and what should be considered.*

2 Building the counter-example

In this section we shall consider different approaches to understand SSE and show that it cannot hold in any of those cases.

2.1 Assuming the existence of the limit extension, which makes SSE well-defined

Here we discuss the extension of limit definition to make SSE work. By the way of contradiction, we assume that it could exist to make the difference of those divergent series to be equal to something. As you get your series with passing to the limit, we shall suppose that in some sense

$$\zeta(s) = \lim_{k\to\infty} \sum_{n=1}^{k} \frac{1}{n^s}, \Re(s) > 0, \tag{4}$$

as it was claimed in your paper before and discussed by Identity theorem. By the section 1.2. and Lemma 1 we deduce that for this extended limit it holds that

$$\zeta(s) = \lim_{k\to\infty} \sum_{n=1}^{m(k)} \frac{1}{n^s}, \tag{5}$$

where $\{m(k)|k \geq 1\}$ is any infinite subsequence of natural numbers (by the definition infinite and strictly increasing). The contradiction is implied from the following claim.

Theorem 3. *Let $x \in \mathbb{R}$ be arbitrary. Then the following identity holds.*

$$\lim_{k\to\infty} [\sum_{n=1}^{\lfloor (k+x)^2 \rfloor} \frac{1}{\sqrt{n}} - \sum_{n=1}^{k^2} \frac{1}{\sqrt{n}}] = 2x.$$

Proof. Assume that x is fixed. Then we consider only k large enough, for which $k + x > 1$. Then by Theorem 2 we obtain two representations for $\zeta(\frac{1}{2})$.

$$\zeta(\frac{1}{2}) = \sum_{n=1}^{\lfloor (k+x)^2 \rfloor} \frac{1}{\sqrt{n}} - 2\sqrt{\lfloor (k+x)^2 \rfloor} + o(1), \tag{6}$$

$$\zeta(\frac{1}{2}) = \sum_{n=1}^{k^2} \frac{1}{\sqrt{n}} - 2\sqrt{k^2} + o(1), \tag{7}$$

where $o(1)$ denotes the term, which converges to zero as $k \to \infty$ (the integral term in Theorem 2 is a remainder of absolutely convergent integral). After substituting (7) to (6) and passing to the limit for $k \to \infty$ we obtain the following.

$$\lim_{k\to\infty} [\sum_{n=1}^{\lfloor (k+x)^2 \rfloor} \frac{1}{\sqrt{n}} - \sum_{n=1}^{k^2} \frac{1}{\sqrt{n}}] = 2 \lim_{k\to\infty} (\sqrt{\lfloor (k+x)^2 \rfloor} - \sqrt{k^2})$$

$$= 2 \lim_{k\to\infty} (\sqrt{\lfloor (k+x)^2 \rfloor} - \sqrt{(k+x)^2} + \sqrt{(k+x)^2} - \sqrt{k^2})$$

$$= 2 \lim_{k\to\infty} (\sqrt{\lfloor (k+x)^2 \rfloor} - \sqrt{(k+x)^2}) + 2 \lim_{k\to\infty} (\sqrt{(k+x)^2} - \sqrt{k^2}) =: L_1 + L_2. \tag{8}$$

Let us calculate L_1 first. We want to prove that it is equal to zero. To obtain this we dominate the absolute value of the general term of this sequence by the general term of some sequence, which is convergent to zero and obtain the result by the squeeze theorem.

$$0 \leq |\sqrt{\lfloor (k+x)^2 \rfloor} - \sqrt{(k+x)^2}| = |\frac{\lfloor (k+x)^2 \rfloor - (k+x)^2}{\sqrt{\lfloor (k+x)^2 \rfloor} + \sqrt{(k+x)^2}}|$$

$$\leq \frac{1}{\sqrt{\lfloor (k+x)^2 \rfloor} + \sqrt{(k+x)^2}} \to 0, k \to \infty. \tag{9}$$

Here we used that the difference of any number with its integer part cannot be greater than one. It remains to calculate L_2.

$$2 \lim_{k\to\infty} (\sqrt{(k+x)^2} - \sqrt{k^2}) = 2 \lim_{k\to\infty} (k + x - k) = 2x. \tag{10}$$

After substituting (9) and (10) into (8) we obtain the theorem statement. □

But if we had the extension for limit definition, which respects the classical limit, by (4) and consequently (5) we would obtain that

$$\forall x \in \mathbb{R} 2x = \lim_{k\to\infty} [\sum_{n=1}^{\lfloor (k+x)^2 \rfloor} \frac{1}{\sqrt{n}} - \sum_{n=1}^{k^2} \frac{1}{\sqrt{n}}] = \zeta(\frac{1}{2}) - \zeta(\frac{1}{2}) = 0,$$

which is obviously a contradiction. Thus, there is no any limit definition, which makes this work. To make the justification complete we should prove that under the limit extension, which preserves analytic continuation, the limits $\lim_{k\to\infty} \sum_{n=1}^{k^2} \frac{1}{n^s}$ and $\lim_{k\to\infty} \sum_{n=1}^{\lfloor (k+x)^2 \rfloor} \frac{1}{n^s}$ generate the same function, despite what you have told me.

Remark 3. *Assume that there exists an extension of the limit definition, which preserves the analytic continuation to the critical strip. Then the following equality holds.*

$$\lim_{k\to\infty} \sum_{n=1}^{k^2} \frac{1}{n^s} = \lim_{k\to\infty} \sum_{n=1}^{\lfloor (k+x)^2 \rfloor} \frac{1}{n^s}, \Re(s) \in (0,1), x \in \mathbb{R}$$

Proof. By the way of contradiction we assume that $\lim_{k\to\infty} \sum_{n=1}^{k^2} \frac{1}{n^s} \neq \lim_{k\to\infty} \sum_{n=1}^{\lfloor (k+x)^2 \rfloor} \frac{1}{n^s}, \Re(s) \in (0,1), x \in \mathbb{R}$. Let us denote $f := \lim_{k\to\infty} \sum_{n=1}^{k^2} \frac{1}{n^s}$ and $g := \lim_{k\to\infty} \sum_{n=1}^{\lfloor (k+x)^2 \rfloor} \frac{1}{n^s}$. By the definition and Lemma 1 we obtain that for $\Re(s) > 1$ we have the following.

$$f(s) = \lim_{k\to\infty} \sum_{n=1}^{k^2} \frac{1}{n^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s) = \lim_{k\to\infty} \sum_{n=1}^{\lfloor (k+x)^2 \rfloor} \frac{1}{n^s} = g(s), \Re(s) > 1.$$

Since $f = g$ on the domain $\Re(s) > 1$, which consists of accumulation points as an open set. Thus, $f = g$ on the critical strip as well. □

2.2 Subtracting those series with a changed order of summation

It is a question, how can one subtract divergent series, when they aren't defined. You have suggested subtraction term-by-term. To make this possible you should assume commutativity of terms of divergent series, i.e., you have to assume that the order of summation does not change the sum. Indeed, in other That is generally untrue and it is justified by the Riemann rearrangement theorem (`https://en.wikipedia.org/wiki/Riemann_series_theorem`). At the point $s = \frac{1}{2}$ your difference appears to be the sum of two series, one of which diverges to $+\infty$ and the other one other one diverges to $-\infty$. But one thing is to say that those permutations exist and the other thing is to actually build those permutations. Note that you can play with those permutations on the entire critical strip. For real values of s it is given by the Riemann rearrangement theorem and for non-real s it is given by Steinitz's theorem (you may find it in the section "Generalisations" by the same link). As you can see, even in complex analysis things appear to be a little bit more complicated ;). Theorem 3 allows us to build an exact permutation to rearrange this series.

Theorem 4. *Let $x \in \mathbb{R}$ be arbitrary. Then the following identity holds.*

$$[\sum_{n=1}^{\lfloor(\max\{2,-\lfloor x\rfloor+1\})^2\rfloor}\frac{1}{\sqrt{n}}-\sum_{n=1}^{(\max\{2,-\lfloor x\rfloor+1\})^2}\frac{1}{\sqrt{n}}]+\sum_{k=\max\{2,-\lfloor x\rfloor+1\}+1}^{\infty}[\sum_{n=\lfloor(k+x)^2\rfloor+1}^{\lfloor(k+1+x)^2\rfloor}\frac{1}{\sqrt{n}}-\sum_{n=k^2+1}^{(k+1)^2}\frac{1}{\sqrt{n}}]=2x.$$

Proof. We note that for any $x \in \mathbb{R}$ we have that $\max\{2, -\lfloor x\rfloor + 1\} + x \geq 1$, so the summation is well-defined and we count each term exactly one time, which makes it a well-defined permutation of the difference of series. Indeed, for $x > -1$ we have that $\max\{2, -\lfloor x\rfloor + 1\} + x > 2 - 1 = 1$ and for $x \leq -1$ we have that $\max\{2, -\lfloor x\rfloor + 1\} + x = -\lfloor x\rfloor + 1 + x \geq 1$ as for negative numbers $-\lfloor x\rfloor + x \geq 0$. As the sum of the series is by definition the limit of partial sums we obtain the following with respect to Theorem 3.

$$[\sum_{n=1}^{\lfloor(\max\{2,-\lfloor x\rfloor+1\}+x)^2\rfloor}\frac{1}{\sqrt{n}}-\sum_{n=1}^{(\max\{2,-\lfloor x\rfloor+1\})^2}\frac{1}{\sqrt{n}}]+\sum_{k=\max\{2,-\lfloor x\rfloor+1\}+1}^{\infty}[\sum_{n=\lfloor(k+x)^2\rfloor+1}^{\lfloor(k+1+x)^2\rfloor}\frac{1}{\sqrt{n}}-\sum_{n=k^2+1}^{(k+1)^2}\frac{1}{\sqrt{n}}]$$

$$=\lim_{l\to\infty}[\sum_{n=1}^{\lfloor(\max\{2,-\lfloor x\rfloor+1\}+x)^2\rfloor}\frac{1}{\sqrt{n}}-\sum_{n=1}^{(\max\{2,-\lfloor x\rfloor+1\})^2}\frac{1}{\sqrt{n}}]+\sum_{k=\max\{2,-\lfloor x\rfloor+1\}+1}^{l}[\sum_{n=\lfloor(k+x)^2\rfloor+1}^{\lfloor(k+1+x)^2\rfloor}\frac{1}{\sqrt{n}}-\sum_{n=k^2+1}^{(k+1)^2}\frac{1}{\sqrt{n}}]$$

$$\lim_{l\to\infty}[\sum_{n=1}^{\lfloor(l+1+x)^2\rfloor}\frac{1}{\sqrt{n}}-\sum_{n=1}^{(l+1)^2}\frac{1}{\sqrt{n}}]=2x.$$

Thus, the proof is complete. □

3 Final Words

3.1 The problems with your subtraction method

For the end of the discussion I want to show, what actually happens, if we restrict ourselves to the permutation you've suggested. Let us use the Theorem 2 and by the similar to yours calculation check if your representation is defined on the critical strip at all. We obtain

$$\zeta(s) = \sum_{n=1}^{b} \frac{1}{n^s} - \frac{b^{1-s}}{1-s} + o(1), \tag{11}$$

$$\zeta(1-\bar{s}) = \sum_{n=1}^{b} \frac{1}{n^{1-\bar{s}}} - \frac{b^{\bar{s}}}{\bar{s}} + o(1). \tag{12}$$

After subtracting (12) from (11) we obtain the following after some obvious transformations of equations and passing to the limit.

$$\lim_{b\to\infty}[\sum_{n=1}^{b} \frac{1}{n^s} - \sum_{n=1}^{b} \frac{1}{n^{1-\bar{s}}}] = \zeta(s) - \zeta(1-\bar{s}) - \lim_{b\to\infty}[\frac{b^{1-s}}{1-s} - \frac{b^{\bar{s}}}{\bar{s}}]$$

$$\zeta(s) - \zeta(1-\bar{s}) - \lim_{b\to\infty} b^{\frac{1}{2}+i\Im(s)}[\frac{b^{\frac{1}{2}-\Re(s)}}{1-s} - \frac{b^{\Re(s)-\frac{1}{2}}}{\bar{s}}] \tag{13}$$

From (13) we can see that your difference is not defined anywhere but the critical strip at all. Indeed, if $\Re(s) > \frac{1}{2}$, then the term $\frac{b^{\Re(s)-\frac{1}{2}}}{\bar{s}}$ blows up to infinity, while $\frac{b^{\frac{1}{2}-\Re(s)}}{1-s}$ tends to zero. As the result, since the difference of those terms is infinite, the multiplication by the exponent, which also diverges to infinity, leaves this divergent. Analogously, if $\Re(s) < \frac{1}{2}$, the term $\frac{b^{\frac{1}{2}-\Re(s)}}{1-s}$ blows up to infinity, the term $\frac{b^{\Re(s)-\frac{1}{2}}}{\bar{s}}$ tends to zero and the similar problem occurs. As the line $\Re(s) = \frac{1}{2}$ is disjoint from the domain $\Re(s) > 1$, by the Remark 1 it has no relation to Riemann zeta-function and cannot be used for the studies of its properties.

3.2 Conclusion

Dear Aric, in the current paper I tried to explain my truth and wanted to show, what you should expect to encounter while defending your approach. I could entirely argue with you on interpretations, but it does not solve the problems. Thing is that your approaches are interesting, but they do not agree with the existing theory. I understand that you do not trust it completely. But still, to build your own theory and invent your own mathematics you should show that it has no contradictions and works well with your set of axioms. You see the current problems and I wish you the best in solving them. I am looking forward to see the beautiful building of your theory, when it is ready!