Proof

Let $\zeta(s)$ be the analytic continuation of the sum $\sum_{n=1}^{\infty} \frac{1}{n^s}$, and let $\zeta(1-s^*)$ be the analytic continuation of the sum $\sum_{n=1}^{\infty} \frac{1}{n^{1-s^*}}$.

We want to show that if $\zeta(s) = \zeta(1-s^*)$, then the difference between the series $\sum_{n=1}^{\infty} \frac{1}{n^s}$ and $\sum_{n=1}^{\infty} \frac{1}{n^{1-s^*}}$ converges to zero in the critical strip, and vice versa.

Assume $\zeta(s) = \zeta(1-s^*)$. By the identity theorem for analytic functions, $\zeta(s)$ and $\zeta(1-s^*)$ are identical on a region that contains the line $\mathrm{Re}(s) = \frac{1}{2}$. Since the series $\sum_{n=1}^{\infty} \frac{1}{n^s}$ and $\sum_{n=1}^{\infty} \frac{1}{n^{1-s^*}}$ converge absolutely for $\mathrm{Re}(s) > 1$, and are analytic in this region, they must be identical in this region. Therefore, the difference between the series,

$$\sum_{n=1}^{\infty} \frac{1}{n^s} - \sum_{n=1}^{\infty} \frac{1}{n^{1-s^*}},$$

converges to 0 as s approaches any point on the line $\mathrm{Re}(s) = \frac{1}{2}$.

Conversely, assume that the difference between the series converges to 0 as s approaches any point on the line $\mathrm{Re}(s) = \frac{1}{2}$. Let $f(s) = \zeta(s) - \zeta(1-s^*)$. Then $f(s)$ is an analytic function on the half-plane $\mathrm{Re}(s) > \frac{1}{2}$. We want to show that $f(s) = 0$ for all s in this half-plane.

By the assumption, the limit of $f(s)$ as s approaches any point on the line $\mathrm{Re}(s) = \frac{1}{2}$ is 0. Since the limit of an analytic function is itself analytic, $f(s)$ is analytic on the half-plane $\mathrm{Re}(s) \geq \frac{1}{2}$. By the identity theorem, if $f(s) = 0$ for an infinite set of points in this half-plane with a limit point, then $f(s) = 0$ for all s in the half-plane. Therefore, it suffices to show that the set of zeros of $f(s)$ has a limit point in the half-plane $\mathrm{Re}(s) \geq \frac{1}{2}$.

Let $z = \frac{1}{2} + it$, where t is a real number. Then

$$f(z) = \zeta\left(\frac{1}{2} + it\right) - \zeta\left(\frac{1}{2} - it\right).$$

Since the series $\sum_{n=1}^{\infty} \frac{1}{n^s}$ converges absolutely for $\mathrm{Re}(s) > 1$, the functions $\zeta(s)$ and $\zeta(1-s^*)$ are well-defined and analytic in the half-plane $\mathrm{Re}(s) > 1$. Therefore, $f(z)$ is well-defined and analytic in the strip $\frac{1}{2} \leq \mathrm{Re}(s) \leq 1$.

By the assumption, $\lim_{s \to z} f(s) = 0$. This means that for any $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(s)| < \epsilon$ whenever $0 < |s - z| < \delta$. Since the function $f(s)$ is analytic in the strip $\frac{1}{2} \leq \mathrm{Re}(s) \leq 1$, the maximum modulus principle implies that $|f(s)|$ attains its maximum on the boundary of this strip. Therefore, we can choose δ small enough so that $|f(s)| < \epsilon$ for all s on the line segments $\mathrm{Re}(s) = \frac{1}{2}$ and $|t| < \delta$. This shows that the zeros of $f(s)$ have a limit point on the line $\mathrm{Re}(s) = \frac{1}{2}$.

By the identity theorem, $f(s) = 0$ for all s in the half-plane $\mathrm{Re}(s) \geq \frac{1}{2}$. Hence, $\zeta(s) = \zeta(1-s^*)$.

Therefore, the difference between the series $\sum_{n=1}^{\infty} \frac{1}{n^s}$ and $\sum_{n=1}^{\infty} \frac{1}{n^{1-s^*}}$ converges to 0 in the critical strip if and only if $\zeta(s) = \zeta(1-s^*)$.