Proof:

Let $\zeta(s)$ be the Riemann zeta function, defined for $\text{Re}(s) > 1$. We want to show that if $\zeta(s) = \zeta(1-s^*)$, then the series difference $\sum_{n=1}^{\infty} \frac{1}{n^s} - \sum_{n=1}^{\infty} \frac{1}{n^{1-s^*}}$ converges to 0 in the critical strip.

First, we consider the functional equation of the zeta function, which relates $\zeta(s)$ and $\zeta(1-s^*)$:

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s^*).$$

Assuming $\zeta(s) = \zeta(1-s^*)$, we have:

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(s).$$

Since $\zeta(s)$ is non-zero for $\text{Re}(s) > 1$, we can divide both sides of the equation by $\zeta(s)$ to obtain:

$$1 = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s).$$

Next, we use the reflection formula for the gamma function:

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}.$$

Substituting $z = 1-s$ into the reflection formula, we get:

$$\Gamma(s)\Gamma(s^*) = \frac{\pi}{\sin(\pi(1-s))} = \pi \csc(\pi s).$$

Substituting this into our previous equation, we have:

$$1 = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \pi \csc(\pi s).$$

Simplifying, we obtain:

$$2^s = \frac{2}{\sin(\pi s)}.$$

Now, we can consider the series difference $\sum_{n=1}^{\infty} \frac{1}{n^s} - \sum_{n=1}^{\infty} \frac{1}{n^{1-s^*}}$:

$$\sum_{n=1}^{\infty} \frac{1}{n^s} - \sum_{n=1}^{\infty} \frac{1}{n^{1-s^*}} = \frac{1}{2^s} \left(\sum_{n=1}^{\infty} \frac{1}{n^s} - \sum_{n=1}^{\infty} \frac{1}{n^{s^*}} \right).$$

Using the relationship $2^s = \frac{2}{\sin(\pi s)}$, we can simplify further:

$$\sum_{n=1}^{\infty} \frac{1}{n^s} - \sum_{n=1}^{\infty} \frac{1}{n^{1-s^*}} = \frac{1}{2^s} \left(\sum_{n=1}^{\infty} \frac{1}{n^s} - \sum_{n=1}^{\infty} \frac{1}{n^{s^*}} \right) = \frac{1}{\sin(\pi s)} \left(\sum_{n=1}^{\infty} \frac{1}{n^s} - \sum_{n=1}^{\infty} \frac{1}{n^{s^*}} \right).$$

By the convergence of the zeta function for $\mathrm{Re}(s) > 1$, both series converge absolutely in this region. Thus, the difference of the series is well-defined.

Now, using the functional equation $\zeta(s) = \zeta(1 - s^*)$, we have shown that $\frac{1}{\sin(\pi s)} = 0$ whenever $\zeta(s) = \zeta(1 - s^*)$. Therefore, the series difference $\sum_{n=1}^{\infty} \frac{1}{n^s} - \sum_{n=1}^{\infty} \frac{1}{n^{1-s^*}}$ converges to 0 in the critical strip.