Assume that $\zeta(s) = \zeta(1-s^*)$ for some complex number s. We want to show that the difference between the series $\sum_{n=1}^{\infty} \frac{1}{n^s}$ and $\sum_{n=1}^{\infty} \frac{1}{n^{1-s^*}}$ converges to zero.

Consider the function $f(z) = \zeta(z) - \zeta(1-z^*)$ and define a contour C in the complex plane. The contour starts at $-iR$, goes along the straight line to iR, makes a small semicircular detour in the upper half-plane to avoid the pole at $z = 1$, and returns along the straight line from iR to $-iR$.

By the residue theorem, we have

$$\oint_C f(z)dz = 2\pi i \operatorname{Res}[f, 1].$$

Since $f(z)$ is analytic except for the poles at $z = 0$ and $z = 1$, we can evaluate the integral along the contour C as

$$\oint_C f(z)dz = \int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz + \int_{R}^{-R} f(x)dx,$$

where C_R denotes the semicircular detour.

As R tends to infinity, the integral along the detour C_R vanishes because $f(z)$ is bounded. Thus, we have

$$\lim_{R\to\infty} \left(\int_{-R}^{R} f(x)dx + \int_{R}^{-R} f(x)dx \right) = 0.$$

Simplifying, we get

$$\int_{-\infty}^{\infty} f(x)dx + \int_{\infty}^{-\infty} f(x)dx = 0.$$

Since $f(z) = \zeta(z) - \zeta(1-z^*)$ and $f(z) = 0$, we have $\zeta(z) = \zeta(1-z^*)$ for all real numbers z.

Now, let's consider the difference between the series $\sum_{n=1}^{\infty} \frac{1}{n^s}$ and $\sum_{n=1}^{\infty} \frac{1}{n^{1-s^*}}$:

$$\sum_{n=1}^{\infty} \left(\frac{1}{n^s} - \frac{1}{n^{1-s^*}} \right) = \sum_{n=1}^{\infty} \frac{n^{s^*} - 1}{n(n^{s^*})} = \sum_{n=1}^{\infty} \frac{a_n}{n},$$

where $a_n = \frac{n^{s^*}-1}{n(n^{s^*})}$.

Since $\zeta(s) = \zeta(1-s^*)$, we know that $\zeta(s)$ converges for $\operatorname{Re}(s) > \frac{1}{2} Now, we can express a_n$ as follows:

$$a_n = \frac{n^{s^*} - 1}{n(n^{s^*})} = \frac{1}{n} - \frac{1}{n^{1-s^*}}.$$

Since $\zeta(s)$ converges for $\operatorname{Re}(s) > \frac{1}{2}$, we have that $\sum_{n=1}^{\infty} \frac{1}{n^s}$ converges absolutely.

Similarly, $\zeta(1-s^*)$ converges for $\operatorname{Re}(1-s^*) > \frac{1}{2}$, which is equivalent to $\operatorname{Re}(s^*) < \frac{1}{2}$. Hence, $\sum_{n=1}^{\infty} \frac{1}{n^{1-s^*}}$ converges absolutely.

Therefore, for any fixed $\mathrm{Re}(s) > \frac{1}{2}$, both series $\sum_{n=1}^{\infty} \frac{1}{n^s}$ and $\sum_{n=1}^{\infty} \frac{1}{n^{1-s^*}}$ converge absolutely.

Since the terms $a_n = \frac{1}{n} - \frac{1}{n^{1-s^*}}$ in the series $\sum_{n=1}^{\infty} \frac{a_n}{n}$ approach zero as n goes to infinity, we conclude that the series $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges to zero.

Therefore, the difference between the series $\sum_{n=1}^{\infty} \frac{1}{n^s}$ and $\sum_{n=1}^{\infty} \frac{1}{n^{1-s^*}}$ converges to zero.

Conversely, if the difference between the series converges to zero, we can reverse the steps of the proof and deduce that $\zeta(s) = \zeta(1 - s^*)$ for the given complex number s.

Hence, we have proven that if $\zeta(s) = \zeta(1 - s^*)$, then the difference between the sums of the series $\sum_{n=1}^{\infty} \frac{1}{n^s}$ and $\sum_{n=1}^{\infty} \frac{1}{n^{1-s^*}}$ must converge to zero. The converse is also true.

This completes the proof.