# Super Symmetric Equation

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Let $\zeta(s)$ be the analytic continuation of the sum of $1/n^s$ from $n = 1$ to $\infty$, and let $\zeta(1 - s^*)$ be the analytic continuation of the sum of $1/n^{1-s^*}$ from $n = 1$ to $\infty$.

Assume $\zeta(s) = \zeta(1 - s^*)$. By the identity theorem for analytic functions, $\zeta(s)$ and $\zeta(1 - s^*)$ are identical on a region that contains the line $\mathrm{Re}(s) = 1/2$. Since the series $\sum_{n=1}^{\infty} 1/n^s$ and $\sum_{n=1}^{\infty} 1/n^{1-s^*}$ converge absolutely for $\mathrm{Re}(s) > 1$, and are analytic in this region, they must be identical in this region. Therefore, the difference between the series, i.e.,

$$\sum_{n=1}^{\infty} \frac{1}{n^s} - \sum_{n=1}^{\infty} \frac{1}{n^{1-s^*}},$$

converges to 0 as $s$ approaches any point on the line $\mathrm{Re}(s) = 1/2$.

Conversely, assume that the difference between the series converges to 0 as $s$ approaches any point on the line $\mathrm{Re}(s) = 1/2$. Let $f(s) = \zeta(s) - \zeta(1 - s^*)$. Then $f(s)$ is an analytic function on the half-plane $\mathrm{Re}(s) > 1/2$. We want to show that $f(s) = 0$ for all $s$ in this half-plane.

By the assumption, the limit of $f(s)$ as $s$ approaches any point on the line $\mathrm{Re}(s) = 1/2$ is 0. Since the limit of an analytic function is itself analytic, $f(s)$ is analytic on the half-plane $\mathrm{Re}(s) \geq 1/2$. By the identity theorem, if $f(s) = 0$ for an infinite set of points in this half-plane with a limit point, then $f(s) = 0$ for all $s$ in the half-plane. Therefore, it suffices to show that the set of zeros of $f(s)$ has a limit point in the half-plane $\mathrm{Re}(s) \geq 1/2$.

Let $z = 1/2 + it$, where $t$ is a real number. Then

$$f(z) = \zeta(z) - \zeta(1 - z^*) = \sum_{n=1}^{\infty} \left( \frac{1}{n^z} - \frac{1}{n^{1-z^*}} \right).$$

Using the fact that $|1/n^z - 1/n^{1-z^*}| \leq 2/n^{1/2}$, we have

$$|f(z)| \leq \sum_{n=1}^{\infty} \frac{2}{n^{1/2}} = 2\zeta(1/2) < \infty.$$

Therefore, by the Cauchy-Schwarz inequality, we have

$$|f(z + ih)| \leq \sqrt{2\zeta(1/2)} \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^{1+h}}} \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^{1-h}}},$$

where $h$ is a small positive real number. Since both $\sum_{n=1}^{\infty} 1/n^{1+h}$ and $\sum_{n=1}^{\infty} 1/n^{1-h}$ converge for $h > 0$, the above inequality shows that $f(z) \to 0$ as $h \to 0$. Therefore, the set of zeros of $f(s)$ has a limit point at $z$ in the half-plane $\mathrm{Re}(s) \geq 1$.